Lecture 9. The concept of monotonic fuctions. Local extremum of the function. The necessary and sufficient conditions of extreme. The asymptote of the function. Research on the graphics of the functions.

Definition 1. A function f is called monotone if for all $x \leq y$ we have $f(x) \leq f(y)$.

An alternative definition is that f is monotone if changing an input bit from 0 to 1 cannot change the value of the function from 1 to 0.

We note that many graph properties (defined as a Boolean functions over Boolean adjacency matrices) are monotone, i.e., adding additional edges cannot destroy the property. Examples of such properties include the graph being connected, containing a clique or having Hamiltonian cycle.

Definition 2. A Boolean circuit is monotone if it contains AND and OR gates only.

It is not hard to see that those two definitions are closely related. If function f is computed by a monotone circuit, then f is monotone, because clearly setting a bit cannot unset the value of any wire. The converse also holds: if f is a monotone function, then there exists a monotone circuit that computes it. Consider all minterms of f and construct the circuit as OR of ANDs of variables in each midterm.

However, there is no guarantee that monotone circuits that compute a monotone function will be of small size, because a monotone function can have exponentially many minterms. This motivates an interesting question: what functions can be computed by monotone circuits of polynomial size?